The Terwilliger algebra of certain spin models(Groups and Combinatorics)

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The Terwilliger algebra of certain spin models.

Brian Curtin
Kyushu University
curtin@math.kyushu-u.ac.jp

The Terwilliger algebra is an algebraic tool for studying association schemes. By a result of F. Jaeger, spin models can be interpreted as association schemes which satisfy a few additional conditions. Thus it is natural to consider the Terwilliger algebra of those association schemes which arise from spin model. The strongest results can be stated for these spin model association schemes which are actually distance-regular graphs.

In this note we will focus on the bipartite distance-regular graphs which are spin models. These graphs are all known due to the work of Nomura. We have also studied these graphs, but from an algebraic perspective. This note is meant to be an elementary examination of some of the algebraic properties of the Terwilliger algebra of these graphs.

1 Distance-regular graphs

Let $\Gamma=(X,R)$ denote a finite, connected, simple graph with diameter D. Γ is said to be *distance-regular* whenever for all integers i, $(0\leq i\leq D)$ and for all x, $y\in X$ with $\partial(x,y)=i$, the numbers

$$\begin{aligned} c_i &= |\{z\,|\,\partial(x,z)=i-1,\ \partial(y,z)=1\}|,\\ a_i &= |\{z\,|\,\partial(x,z)=i,\ \partial(y,z)=1\}|,\\ b_i &= |\{z\,|\,\partial(x,z)=i+1,\ \partial(y,z)=1\}| \end{aligned}$$

are independent of x and y. The constants c_i, a_i, and b_i $(0\leq i\leq D)$ are known as the *intersection numbers* of Γ.

For each integer i $(0\leq i\leq D)$, let A_i denote the matrix with x, y entry

$$(A_i)(x,y)=\begin{cases}1 & \text{if } \partial(x,y)=i,\\ 0 & \text{otherwise}\end{cases}\qquad (x,y\in X).$$

The matrix A_i is known as i^{th}*-distance matrix for* Γ. ($A=A_1$ is the *adjacency matrix.*) Let M denote the complex matrix algebra

$$M=<A>.$$

The algebra M is called the *Bose-Mesner algebra of* Γ. It is a well known fact that M has basis A_0, A_1, $\ldots$, A_D. For more details about distance-regular graphs we refer the reader to [1] or [2].

2 Spin models

Let X be a finite nonempty set of size n. A *spin model* is a matrix W whose rows and columns are indexed by X with nonzero entries which satisfies the following equations for all a, b, $c \in X$:

$$\begin{aligned} \sum_{x\in X} W(x,b)W(x,c)^{-1} &= n\delta_{b,c}, \\ \sum_{x\in X} W(x,a)W(x,b)W(x,c)^{-1} &= \sqrt{n}W(a,b)W(a,c)^{-1}W(c,b)^{-1}. \end{aligned}$$

For all b, $c \in X$, define the column vector Y_{bc} by

$$Y_{bc}(x) = \frac{W(x,b)}{W(x,c)} \qquad (x \in X).$$

Then $N(W)$ is defined to be the set of all matrices A such that, for all b, $c \in X$, the vector Y_{bc} is an eigenvector of A.

It turns out that $N(W)$ is the Bose-Mesner algebra of some association scheme and that $W \in N(W)$.

Let $\Gamma = (X, R)$ be a distance-regular graph, and let M denote the Bose-Mesner algebra of Γ. A spin model W is said to be *supported by* Γ whenever $W \in M \subseteq N(W)$. For more details on spin models and the facts quoted here we refer the reader to [6] (or [8]).

3 The Terwilliger algebra

Fix any $x \in X$. We write

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}.$$

For each integer i $(0 \le i \le D)$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbf{C})$ with y,y entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } y \in \Gamma_i(x), \\ 0 & \text{otherwise} \end{cases} \qquad (y \in X).$$

Let $T = T(x)$ denote the subalgebra of $Mat_X(\mathbf{C})$ generated by M and E_0^*, $E_1^*, \ldots, E_D^*$. The algebra T is called the *Terwilliger algebra of* Γ *with respect to* x.

Define operators $L = L(x)$, $F = F(x)$, $R = R(x)$:

$$L = \sum_{h=0}^{D} E_{h-1}^* A E_h^*, \qquad F = \sum_{h=0}^{D} E_h^* A E_h^*, \qquad R = \sum_{h=0}^{D} E_{h+1}^* A E_h^*.$$

Then

$$A = L + F + R.$$

We refer to L, F, and R as the *lowering matrix*, the *flat matrix*, and the *raising matrix* with respect to x, respectively.

T is a semisimple algebra over $\mathbf{C}$, so T decomposes into the direct sum of full matrix algebras:

$$T = \bigoplus_{i=0}^{s} T_i, \qquad T_i \cong Mat_{n_i}(\mathbf{C}).$$

For all i $(0 \leq i \leq s)$, let φ_i denote the orthogonal projection of T onto T_i. For more details on semisimple algebra, see for example [5].

By Terwilliger [9], for every i $(0 \leq i \leq s)$ there exist numbers $r(i)$, $d(i)$ such that

$$E_j^* \varphi_i = 0 \Leftrightarrow j < r(i) \text{ or } j > r(i) + d(i).$$

The numbers $r(i)$ and $d(i)$ are called the *endpoint* and *diameter* of T_i, respectively.

T_i is said to be *thin* if $E_j^* \varphi_i$ has rank at most 1 for all j $(0 \leq i \leq D)$. Γ is said to be *thin* if T_i is thin for all i $(0 \leq i \leq s)$.

The bipartite distance-regular graphs are easily described.

Lemma 1 *(see for example [3]) Let $\Gamma = (X, R)$ be a distance-regular graph of diameter D. Then the following are equivalent.*

(i) Γ *is* bipartite.

(ii) $a_i = 0$ *for all* i $(0 \leq i \leq D)$.

(iii) $F = 0$. □

4 Background

It turns out that the bipartite distance-regular graphs which support a spin model are the 2-homogeneous. Thus we will focus on this combinatorial property.

Definition 2 Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph of diameter $D \geq 3$ and valency $k \geq 3$. Γ is said to be *2-homogeneous* whenever for all integers i $(1 \leq i \leq D-1)$ and all x, y, $z \in X$ with $\partial(y, z) = 2$, $\partial(x, y) = i$, $\partial(x, z) = i$, the number

$$\gamma_i = |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i-1}(x)|$$

is independent of x, y, z.

Theorem 3 *(Nomura [7]) Any bipartite distance-regular graph which supports a spin model is 2-homogeneous.*

Theorem 4 *(Curtin [4]) Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Suppose Γ is 2-homogeneous. Then there exists a real scalar q such that*

$$
\begin{aligned}
c_i &= \frac{(q^D + q^2)}{(q^D + q^{2i})} \frac{(q^{2i} - 1)}{(q^2 - 1)} & (0 \leq i \leq D),\\
b_i &= \frac{(q^D + q^2)(q^D - q^{2i-D})}{(q^D + q^{2i})(q^2 - 1)} & (0 \leq i \leq D)\\
&= c_{D-i},\\
\gamma_i &= \frac{(q^D + q^2)}{(q^D + q^4)} \frac{(q^D + q^{2i+2})}{(q^D + q^{2i})} & (1 \leq i \leq D-1),
\end{aligned}
$$

where we allow the limiting cases $q \mapsto \pm 1$.

This parameterization is equivalent to the 2-homogeneous property.

5 The operators R and L

Lemma 5 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph of diameter $D \geq 3$ and valency $k \geq 3$. Fix $x \in X$. Then the following are equivalent.*

(i) *Γ is 2-homogeneous.*

(ii) *There exist scalars γ_i $(1 \leq i \leq D-1)$ such that*

$$E_i^* L R E_i^* = b_i E_i^* + (\mu - \gamma_i) E_i^* A_2 E_i^*.$$

(iii) *There exist scalars γ_i $(1 \leq i \leq D-1)$ such that*

$$E_i^* R L E_i^* = c_i E_i^* + \gamma_i E_i^* A_2 E_i^*.$$

(iv) *RLE_i^*, LRE_i^*, E_i^* are linearly dependent.*

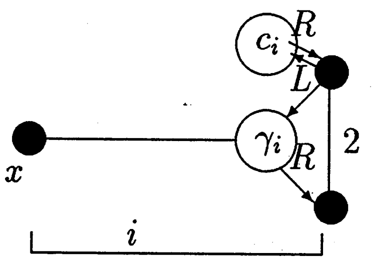
Suppose (i) – (iv) hold. Then the scalars γ_i of 2-homogeneous and (ii), (iii) are all equal.

Proof. (sketch)

(i)$\Rightarrow$(iii) For vertices y, $z \in \Gamma_i(x)$ consider the y, z entry of RL.

$$RL(y,z) = \begin{cases} c_i & \text{if } y = z, \\ \gamma_i & \text{if } \partial(y,z) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix form of this observation is (iii). Pictorially, we have:



(iii)$\Rightarrow$(i) Consider the y, z entry of each side when y, $z \in \Gamma_i(x)$ and $\partial(y,z) = 2$.

(i)$\Leftrightarrow$(ii) Similar.

(ii), (iii)$\Rightarrow$(iv) Clear

(iv)$\Rightarrow$(i) Straight forward. □

Observe that $0 < \gamma_i < \mu$ for all i $(1 \leq i \leq D-1)$.

Lemma 6 *Let $\Gamma = (X, R)$ denote a 2-homogeneous bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix $x \in X$. Then*

$$E_i^* LRE_i^* = \sigma_i E_i^* RLE_i^* + \rho_i E_i^*,$$

where

$$\begin{aligned} \sigma_i &= \frac{(q^{D+2} + q^{2i})}{(q^D + q^{2i+2})}, \\ \rho_i &= \frac{(q^D + q^2)^2 (q^D - q^{2i})}{q^D (q^2 - 1)(q^D + q^{2i+2})}. \end{aligned}$$

Proof. By scaling, we can choose nonzero elements $v_j \in \varphi_i E^*_{r(i)+j} T$ $(0 \leq j \leq d(i))$ such that

$$Rv_j = v_{j+1}.$$

We show that this is a basis for T_i and describe the action of L on this basis by induction. Observe that $Lv_0 = 0$ by the definition of endpoint. Suppose that

$$Lv_j = \chi_{j-1}(i) v_{j-1},$$

where $\chi_{-1}(i) = 0$.

Now

$$\begin{aligned} Lv_{j+1} &= LRE^*_{i+j}v_j \\ &= \sigma_{i+j}RLE^*_{i+j}v_j + \rho_{i+j}E^*_{i+j}v_j) \\ &= \sigma_{i+j}\chi_{j-1}(i)Rv_{j-1} + \rho_{i+j}v_j \\ &= (\sigma_{i+j}\chi_{j-1}(i) + \rho_{i+j})v_j. \end{aligned}$$

In particular, for every i, Lv_{j+1} is a multiple of v_j. We define $\chi_j(i)$ to be the solution to the recurrence

$$\begin{aligned} \chi_j(i) &= \sigma_{i+j}\chi_{j-1}(i) + \rho_{i+j} \\ \chi_{-1}(i) &= 0. \end{aligned}$$

It is now routine to verify that

$$\chi_j(i) = \frac{(q^D + q^2)^2(q^{2j+2} - 1)(q^D - q^{4i+2j-D})}{(q^2 - 1)^2(q^D + q^{2i+2j})(q^D + q^{2i+2j+2})}$$

is the solution to the recurrence. □

Corollary 7 *Referring to Lemma 6:*

(i) *Γ is thin.*

(ii) *For every r $(0 \leq r \leq \lfloor D/2 \rfloor)$ there is a unique T_i with $r(i) = r$.*

(iii) *The diameter of T_i is $d(i) = D - 2r(i)$.*

Proof. (sketch)

(i) Observe that the v_j form a basis.

(ii) Observe that the numbers χ_j only depend upon i.

(iii) This is a lower bound for $d(i)$ by Terwilliger ([9]), and this is an upper bound since $\chi_{D-2r(i)+1}(j) = 0$. □

Corollary 8 *Let T_r be the unique block with endpoint r. Then T_r has basis $v_0, v_1, \ldots, v_{D-2r}$ such that*

$$\begin{aligned} Lv_j &= b_{j-1}(r)v_{j-1}, \\ Rv_j &= c_{j+1}(r)v_{j+1}, \end{aligned}$$

where

$$\begin{aligned} c_j(r) &= \frac{(q^r(q^2 + q^D)(q^2j - 1)}{(q^D + q^{2j+2r})(q^2 - 1)}, \\ b_j(r) &= \frac{(q^D + q^2)(q^D - q^{4r+2j-D})}{q^r(q^D + q^{2r+2j})(q^2 - 1)}. \end{aligned}$$

Proof. This is just a rescaling of the basis of Lemma 6. Observe that it preserves $LRv_j = \chi_{j-1}(r)v_j$, $Lv_0 = 0$, and $Rv_{d(r)+r} = 0$. $\square$

A simple consequence of this corollary is the following.

Theorem 9 *The Terwilliger algebra of any 2-homogeneous bipartite distance-regular graph is a quantum Lie algebra with respect to the operators L, F, and R.*

We conjecture that the Terwilliger algebra of any distance-regular graph which supports a spin model has a similar structure.

Let us conclude with an observation about the numbers $c_i(r)$ and $b_i(r)$.

Lemma 10 *Fix r $(0 \leq r \leq d)$. Let θ_r be the r^{th} eigenvalue, and write $d = d(r) = D - 2r$.*

$$\begin{aligned} b_0(r) &= \theta_r, \\ b_i(r) + c_i(r) &= b_0(r), \\ b_{d-i}(r) &= c_i(r), \\ c_0(r) &= 0, \\ b_d(r) &= 0. \end{aligned}$$

Proof. (sketch)

Compare the various formulas in q for the quantities involved. $\square$

These conditions satisfied by the intersection numbers with $d = D$. In fact, $c_j(0) = c_j$, $b_j(0) = b_j$.

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